Equating co-efficient of $\cos\theta$,

$U\cos\theta = KA_K a^{-(K+1)} \cos K\theta \Rightarrow k=1 \Rightarrow U\cos\theta = 1 A_1 a^{-(1+1)} \cos\theta$

$U = A_1 a^{-2}$

$Ua^2 = A_1$.

Equating co-efficient of $\cos(n+1)\theta$,

$U\epsilon/_2 \cos(n+1)\theta + Un\,\epsilon/_2 (\cos(n+1)\theta)$
$KA_K a^{-(K+1)} (-\epsilon/_2 (n+K)\cos(n+K)\theta) + KA_K a^{-(K+1)} \cos K\theta$

$\Downarrow K=1 \qquad \Downarrow k=n+1$

$= A_1 a^{-2} \left(-\epsilon/_2 (n+1)\cos(n+1)\theta\right) + (n+1) A_{n+1}$

$a^{-(n+2)} \cos(n+1)\theta$.

$U\epsilon/_2 + Un\,\epsilon/_2 = -A_1 a^{-2} \epsilon/_2 (n+1) + (n+1) A_{n+1} a^{-(n+2)}$

$U\epsilon/_2 \cancel{(n+1)} = -A_1 a^{-2} \epsilon/_2 \cancel{(n+1)} + \cancel{(n+1)} A_{n+1}$

$a^{-(n+2)}$

$U\epsilon/_2 = -Ua^2 \cancel{a^{-2}} \epsilon/_2 + A_{n+1} a^{-(n+2)}$

$U\epsilon/_2 + U\epsilon/_2 = A_{n+1} a^{-(n+2)}$ [$\because$ sub the value of A. (ie) $A_1 = Ua^2$]

$U\epsilon = A_{n+1} a^{-(n+2)}$

$\Rightarrow A_{n+1} = U\epsilon\, a^{-(n+2)}$

Equating co-efficient of $\cos(n-1)\theta$,

$$U\varepsilon/_2 \cos(n-1)\theta - Un\,\varepsilon/_2 \cos(n-1)\theta \quad \pi\ k=1 \quad +KA_K a^{-(K+1)}\cos K\theta$$

$$KA_K a^{-(K+1)}(\varepsilon/_2 (n-K)\cos(n-K)\theta$$

$$= A_1 a^{-2}\,\varepsilon/_2\,(n-1)\cos(n-1)\theta + (n-1)A_{n-1}a^{-(n-1)}$$

$\cos(n-1)\theta$.

$$U\varepsilon/_2 - Un\,\varepsilon/_2 = A_1 a^{-2}\,\varepsilon/_2\,(n-1) + (n-1)A_{n-1}a^{-n}$$

$$-U\varepsilon/_2\,(n-1) = A_1 a^{-2}\,\varepsilon/_2\,(n-1) + (n-1)A_{n-1}a^{-n}$$

$$-U\varepsilon/_2 = Ua^2 a^{-2}\,\varepsilon/_2 + A_{n-1}a^{-n} \qquad \{\because A_1 = Ua^2\}$$

$$-U\varepsilon/_2 - U\varepsilon/_2 = A_{n-1}a^{-n}$$

$$\Rightarrow -2U\varepsilon/_2 = A_{n-1}a^{-n} \Rightarrow -U\varepsilon = A_{n-1}a^{-n}$$

$$\Rightarrow A_{n-1} = -U\varepsilon a^{n}.$$

$$A_{n-1} = -U\varepsilon/a^{-n}$$

$$\Rightarrow A_{n-1} = -U\varepsilon a^{n}.$$

All other A's are zero substitution for

A_1, A_{n-1}, A_{n+1} yields the required result.

$$q_R = \sum_{K=1}^{\infty} KA_K R^{-(K+1)}\cos K\theta \qquad R = a(1+\varepsilon\cos n\theta)$$

$$q_R = \sum_{K=1}^{\infty} KA_K\,[a(1+\varepsilon\cos n\theta)]^{-(K+1)}\cos K\theta.$$

$K = 1, n+1, n-1$

$$q_R = A_1[a(1+\varepsilon\cos n\theta)]^{-2}\cos\theta + (n+1)A_{n+1}[a(1+\varepsilon\cos n\theta)]^{-(n+2)}\cos(n+1)\theta + (n-1)A_{n-1}[a(1+\varepsilon\cos n\theta)]^{-n}\cos(n-1)\theta.$$

$A_1 = Ua^2, \quad A_{n+1} = U\varepsilon a^{n+2}, \quad A_{n-1} = -U\varepsilon a^n$

$$q_R = Ua^2 R^{-2} \cos\theta + (n+1) U\varepsilon a^{n+2} R^{-(n+2)}$$

$$\cos(n+1)\theta + (n-1)(-U\varepsilon a^n) R^{-n} \cos(n-1)\theta.$$

$$q_R = Ua^2 R^{-2} \cos\theta + (n+1) Ua^{n+2} \varepsilon R^{-(n+2)}$$

$$\cos(n+1)\theta - (n-1) U\varepsilon a^n R^{-n} \cos(n-1)\theta.$$

WKT,

$$q_R = -\partial\phi/\partial R.$$

$$\frac{-\partial\phi}{\partial R} = \underbrace{Ua^2 R^{-2}}_{\Rightarrow 1/R^2} \cos\theta + (n+1) U\varepsilon a^{n+2} \underbrace{R^{-(n+2)}}_{\Rightarrow 1/R^{n+2}}$$

$$\cos(n+1)\theta - (n-1) U\varepsilon a^n \underbrace{R^{-n}}_{\Rightarrow 1/R^n} \cos(n-1)\theta$$

$\int$ing w.r.to 'R', $\int 1/x^2 = x^{-2} = \frac{x^{-2+1}}{-2+1} = \frac{x^{-1}}{-1}$

$$-\phi = Ua^2 \cos\theta \frac{R^{-1}}{-1} + (n\cancel{+1}) U\varepsilon a^{n+2} \frac{R^{-(n+1)}}{-(n\cancel{+1})}$$

$$\cos(n+1)\theta - (n\cancel{-1}) U\varepsilon a^n \cos(n-1)\theta \frac{R^{-(n-1)}}{-(n\cancel{-1})}$$

$$-\phi = -\frac{Ua^2\cos\theta}{R} - \frac{U\varepsilon a^{n+2}\cos(n+1)\theta}{R^{n+1}}.$$

$$+\frac{U\varepsilon a^n \cos(n-1)\theta}{R^{n-1}}$$

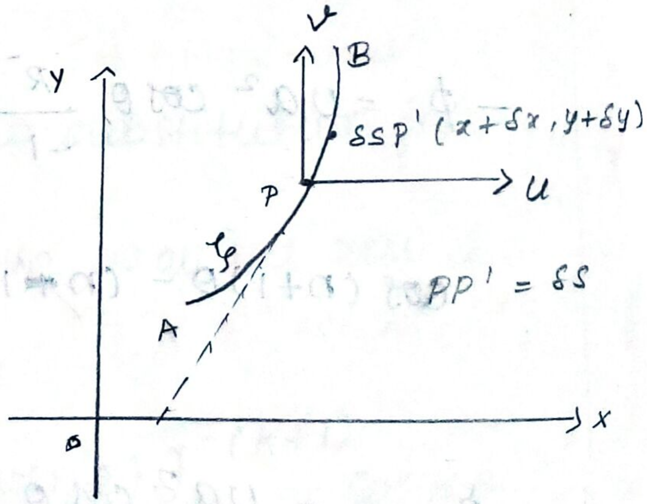
$$\phi = \frac{Ua^2 \cos\theta}{R} + \frac{U\varepsilon a^{n+2}\cos(n+1)\theta}{R^{n+1}} - \frac{U\varepsilon a^n \cos(n-1)\theta}{R^{n-1}}$$

$$\phi = Ua\left\{ (a/R)\cos\theta + \varepsilon (a/R)^{n+1}\cos(n+1)\theta - \varepsilon (a/R)^{n-1}\cos(n-1)\theta \right\}$$

Hence the proof.

STREAM FUNCTIONS :-

C is an arc of a curve in the xy-plane joining the two points A, B.



Fluid (compressible (or) incompressible) is constrained to flow in two-dimensional steady motion parallel to this plane.

The velocity component at P are $[u, v]$.

P' is the neighbouring points $(x + \delta x, y + \delta y)$.

such that, arc $PP' = \delta s$.

Let the tangent at P makes an angle α.

The velocity component at P along the normal from right to left has be travel from A to B is,

$(v \cos\alpha - u \sin\alpha)$.

The mass of the fluid which crosses unit thickness of the surface element through PP' normal to the plane of flow per unit time is,

Mass $\rightarrow \rho(v \cos\alpha - u \sin\alpha)\delta s$.

Mass = density × volume

volume = velocity × area

$= \rho\,[v\,\delta x - u\,\delta y]$.

$(v\cos\alpha - u\sin\alpha) \times \delta s$

$uv \rightarrow$ propotional to each other

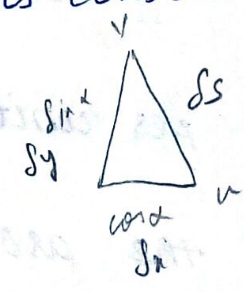
$u \rightarrow$ horizontal $\rightarrow \cos\alpha$

$v \rightarrow$ vertical $- \sin\alpha$

$\frac{u}{v} = \frac{\cos\alpha}{\sin\alpha}$

$(v\cos\alpha - u\sin\alpha)$

This is the mass flux which is constant and is denoted by $\delta\psi$.



$\sin\alpha = \frac{\delta y}{\delta s}$

$= \rho\left[v\frac{\delta x}{\delta s} - u\frac{\delta y}{\delta s}\right)\delta s$

$\cos\alpha = \frac{adj}{hyp} = \frac{\delta x}{\delta s}$

mass flux

$\therefore \delta\psi = \rho\,[v\,\delta x - u\,\delta y]$

$d\psi = \rho\,[v\,dx - u\,dy] \rightarrow (1)$

clearly when A and B are fixed and integral (2) is independent of the shape of curve C joining them.

when the flow is steady the eqn of the continuity becomes,

Steady flow
$\nabla\cdot(\rho\vec{q}) = 0$
Incompressible
$\nabla\cdot q = 0$

$\nabla\cdot(\rho\vec{q}) = 0$ ($\vec{q} \Rightarrow (u, v)$)

$\nabla \Rightarrow \partial/\partial x + \partial/\partial y$

$\partial/\partial x\,(\rho u) + \partial/\partial y\,(\rho v) = 0 \rightarrow (3)$

Differentiating partially w.r.to x and y we have, eqn (1) becomes

$\Rightarrow d\psi = \rho\,[v\,dx - u\,dy]$

$$\frac{\partial\psi}{\partial x} = \rho v \; ; \; \frac{\partial\psi}{\partial y} = -\rho u \rightarrow (*)$$

equation (1) after,

[The total mass flux per unit thickness per unit time from right to left across the arc AB is given by,

$$\int_A^B d\psi = \int_A^B \rho\,[v\,dx - u\,dy]$$

$$[\psi]_A^B = \int_A^B \rho\,[v\,dx - u\,dy].$$

$$\psi = \psi_B - \psi_A = \int_A^B \rho\,[v\,dx - u\,dy] \rightarrow (2')$$

suppose the curve C is a stream line, then no fluid crosses in the boundary.

$\therefore$ From A to B.

$\psi_B - \psi_A = 0.$

(or)

ψ = constant along C { $\because$ ϕ - equi-potential, ψ - stream lines

Hence it follows that the family of curves ψ = constant stream lines in the plane $z=0$.

The function $\psi(x,y)$ is called the stream function.

When the fluid is incompressible the equation of continuity $\nabla \cdot \vec{q} = 0.$

(even if the flow pattern is unsteady)

Let $\delta\psi$ denote the volume of the fluid crossing unit thickness of PP' from right to left per unit time. Then we have above equations in the form.

$$d\psi = v\,dx - u\,dy.$$

$$\psi_B - \psi_A = \int_A^B v\,dx - u\,dy.$$

$$\nabla \cdot \vec{q} = 0.$$

$$③ \Rightarrow \partial/\partial x\,(u) + \partial/\partial y\,(v) = 0$$

$$\partial\psi/\partial x = v, \quad \partial\psi/\partial y = -u \longrightarrow ④$$

Suppose we assume further that the flow is irrotational. Then there exists a velocity potential ϕ.

$$\Rightarrow \vec{q} = -\nabla\phi \Rightarrow -\left(\frac{\partial\phi}{\partial x} + \frac{\partial\phi}{\partial y}\right)$$

such that,

$$u = -\partial\phi/\partial x, \quad v = -\partial\phi/\partial y \longrightarrow ⑤$$

The equation of continuity is,

$$\nabla \cdot \vec{q} = 0.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

comparing equations ④ and ⑤.

we have,

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \phi}{\partial y} = -\partial \psi / \partial x$$

Now, Diff 'x' $\Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y} \Rightarrow \frac{\partial^2 \phi}{\partial y^2} = \frac{-\partial^2 \psi}{\partial x \partial y}$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = +\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y}.$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\Rightarrow \phi$ is Harmonic

Illly,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\frac{\partial^2 \phi}{\partial x \partial y} + \frac{\partial^2 \phi}{\partial x \partial y}$$

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$\Rightarrow \psi$ is harmonic.

Also,

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y}=\left(\frac{\partial\psi}{\partial y}\cdot\frac{\partial\psi}{\partial x}\right)+\left(-\frac{\partial\psi}{\partial x}\cdot\frac{\partial\psi}{\partial y}\right)$$

$$=0.$$

Thus for irrotational, incompressible 2-dimensional flow (compressible (or) incompressible).

$\psi(x,y)$, $\phi(x,y)$ are harmonic functions, and the famillies of the curve.

ϕ = constant (equipotentials)

and

ψ = constant (stream lines)

intersect orthogonally.

The last condition is follows from eq

(*).

which states that, the vectors $\nabla\phi$, $\nabla\psi$

are perpendicular.

THE COMPLEX POTENTIAL FOR 2-DIMENSIONAL IRROTATIONAL INCOMPRESSIBLE FLOW:

suppose that $z = x + iy$ and

$$w = f(z) = f(x+iy)$$

$$\Rightarrow w = \phi(x,y) + i\,\psi(x,y).$$

$i^2 = -1$

where, x, y, ϕ, ψ are all real and $i = \sqrt{(-1)}$

we write,

$$\phi = Re(w)$$

$$\psi = img(w)$$

we suppose that ϕ and ψ and their first derivative are contenuous everywhere with in the region.

If at any point of the region the derivative,

$\frac{dw}{dz} = f'(z)$ is unique, then w is said to be analytic (or) regular at that point.

If the derivative is unique throughout the region, then w is said to be analytic or regular throughout the region.

By definition, $f'(z) = \lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right)$

$$\frac{dw}{dz} = f'(z) = \lim_{\delta x, \delta y \to 0} \left(\frac{\delta \phi + i \delta \psi}{\delta x + i \delta y} \right)$$

where,

$$\delta \phi = \phi(x + \delta x, y + \delta y) - \phi(x, y)$$

$$\delta \psi = \psi(x + \delta x, y + \delta y) - \psi(x, y)$$

The limit is evaluated for 1st keeping and y = constant $\Rightarrow \delta y = 0$.

$$f'(z) = \lim_{\delta x \to 0} \left[\frac{\delta \phi + i \delta \psi}{\delta x} \right] = \frac{\partial \phi}{\partial x} + i \frac{\partial \psi}{\partial x} \quad \text{(I)}$$

Illy,

keeping x = constant $\Rightarrow \delta x = 0$.

$$f'(z) = \lim_{\delta y \to 0} \left[\frac{\delta\phi + i\,\delta\psi}{i\,\delta y} \right] = -i\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y}$$

$$\Rightarrow f'(z) = \frac{\partial\psi}{\partial y} - i\frac{\partial\phi}{\partial y} \rightarrow \text{II}$$

$$\text{I} = \text{II}.$$

Equating real and imaginary part.

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}.$$

$$\frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

This equations is called cauchy Riemann equations. It can be shown that the C.R equations are necessary and sufficient condition for w to be analytic at a point function.

The function ϕ, ψ are termed as a "conjugate functions."

Let us now assign ϕ, ψ as velocity potential and stream function for 2-

dimensional, irrotational inviscid flow.

Then, $w = \phi + i\psi$ is termed as "complex velocity potential".

The real and imaginary parts of w are velocity potential (ϕ) and stream function (ψ) respectively.

$$\frac{dw}{dz} = -u + iv.$$

where,

$$u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$$

and

$$\left| \frac{dw}{dz} \right| = \sqrt{(-u^2)^2 + v^2} = q$$

which is the speed of the fluid at any point.

PROBLEM : 1

Discuss the flow for which $w = z^2$.

Solution :-

Given $w = z^2 \rightarrow (1)$

W.K.T,

$w = \phi + i\psi$.

$(1) \Rightarrow \phi + i\psi = (x+iy)^2$.

$\phi + i\psi = x^2 - y^2 + 2ixy$

$\Rightarrow \phi = x^2 - y^2$.

and $\psi = 2xy$.

The equi-potentials are given by,

ϕ = constant.

(ie) $x^2 - y^2$ = constant.

$\Rightarrow x^2/a^2 - y^2/a^2 = 2$

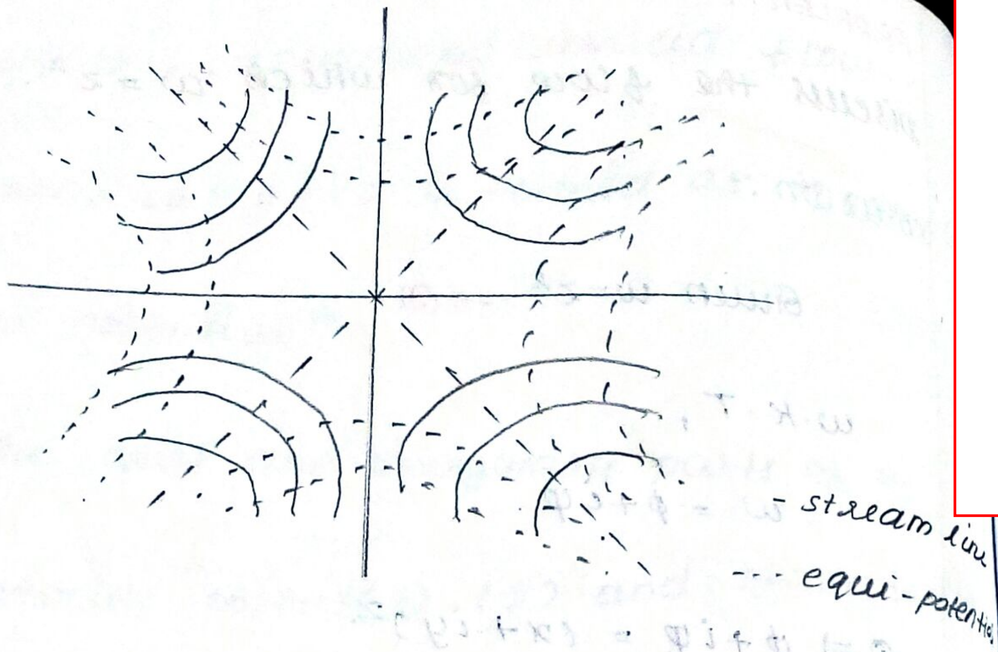
which is a rectangular hyperbola.

IIIly,

ψ = constant

$\Rightarrow xy$ = constant

which is again a rectangular hyperbola.



* The two families of curves cut each other orthogonally in the figure.

* since there is no flow over a stream line any one may be taken as a rigid barrier.

* The axes $x=0, y=0$ may be taken as rigid boundaries.

* since $\frac{dw}{dz} = 2z$ is zero only at the origin and this is the only stagnation point.

ϕ, ψ are harmonic functions and the flow is irrotational.

COMPLEX VELOCITY POTENTIAL FOR STANDARD 2-DIMENSIONAL FLOWS.

(i) uniform stream.

consider the uniform stream having velocity $-U\hat{i}$.

The velocity potential ϕ is given by as follows,

$$q = -u\hat{i}$$

$$\vec{q} = -\nabla\phi \qquad \Rightarrow -U\hat{i} = -\left[i \frac{\partial \phi}{\partial x} \right].$$

$$-U = -\frac{\partial \phi}{\partial x}$$

$$Ux = \phi.$$

The complex velocity potential is given by.

$$w = \phi + i\psi$$

$$w = Uz$$

$$w = U(x + iy)$$

$$\overset{\phi + i\psi}{w} = Ux + iUy.$$

$$\phi = Ux, \ \psi = Uy$$

suppose the uniform stream is incident to the positive x-axis at angle α.

Then,

$$\vec{q} = [-U\cos\alpha, -U\sin\alpha]$$

The velocity potential ϕ is given by,

$$\overset{(-U\cos\alpha, -U\sin\alpha)}{\vec{q}} = -\nabla\phi \Rightarrow -\left[i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y}\right]$$

Hence, $$\frac{\partial\phi}{\partial x} = U\cos\alpha$$

$$\frac{\partial\phi}{\partial y} = U\sin\alpha$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy$$

$$d\phi = U\cos\alpha\, dx + U\sin\alpha\, dy$$

$$\phi = Ux\cos\alpha + Uy\sin\alpha$$

$$\phi = U(x\cos\alpha + y\sin\alpha).$$

using C.R equation, $\Rightarrow U_x = V_y$

$U_y = -V_x$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = U \cos\alpha.$$

$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = -U \sin\alpha.$$

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy \Rightarrow -U \sin\alpha\, dx + U \cos\alpha\, dy$$

$$\psi = Uy \cos\alpha - Ux \sin\alpha$$

$$\psi = U [y \cos\alpha - x \sin\alpha]$$

$$w = \phi + i\psi$$

$$w = U[x \cos\alpha + y \sin\alpha] + i U [y \cos\alpha - x \sin\alpha]$$

$$= U [\cos\alpha (x+iy) + \sin\alpha (y - ix)]$$

$$= U [\cos\alpha (x+iy) - i \sin\alpha (x+iy)].$$

$$= U [z \cos\alpha - iz \sin\alpha]$$

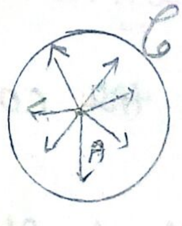
$$= Uz [\cos\alpha - i \sin\alpha]$$

$$\therefore W = Uz e^{-i\alpha}.$$

flow radially outward inwarded from all directions

LINE SOURCE AND LINE SINKS:

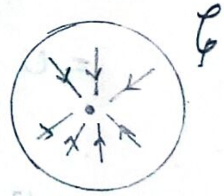
Let A be any point in the plane of flow and C is any closed curve surrounding it



Let parallel infinite lines be drawn through A and through every point of C so that all are $\perp^r$ to the plane of flow.

Suppose the fluid is emitted symmetrically from all points of the infinite line through A, the rate of emission is same everywhere and parallel to the plane of flow, then the line through A is called a <u>line source</u>.

If the fluid drains away through such a line and



under the same conditions of symmetry then the line is called a <u>line sinks</u>.

PROBLEM : 2 :

Find the complex velocity potential of a line source of strength m.